ASYMPTOTIC SOLUTION OF THE ELASTICITY PROBLEM FOR A HOLLOW, FINITE LENGTH, THIN CYLINDER

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The state of stress of a finite-length, hollow cylinder subjected to an axisymmetric load distributed over its entire surface is investigated. The case of a relatively thin cylinder is studied. The accuracies of existing applied theories are examined using the three-dimensional solution as a basis. A method of constructing more accurate solutions is given.

1. Solutions of the homogeneous equations for a hollow cylinder. Consider the axisymmetric deformation of a hollow, isotropic cylinder bounded by coaxial circular cylindrical surfaces having radii R_1 and R_2 and by the planes $z = l$ and $z = -l$ (see Fig.1). Initially, assume that the cylinder is loaded only on the end faces, Γ_1. In terms of displacements, the equilibrium equations are

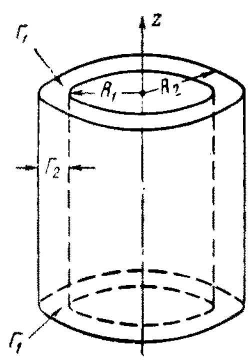

Fig. 1

$$\frac{1}{1-2\nu}\frac{\partial\theta}{\partial z}+\Delta w=0, \qquad \frac{1}{1-2\nu}\frac{\partial\theta}{\partial r}+\Delta u-\frac{1}{r^2}u=0 \tag{1.1}$$

Here

$$\theta=\frac{\partial w}{\partial z}+\frac{\partial u}{\partial r}+\frac{u}{r}, \qquad \Delta=\frac{\partial^2}{\partial z^2}+\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}$$

Consider the solutions of the homogeneous equations of system (1.1), i.e. solutions in the absence of any loading on the cylindrical surfaces $r = R_1$ and R_2. These solutions, which were first obtained in [1], may be found by setting (*)

$$u=a(r)\,dm/dz, \qquad w=b(r)\,m(z) \tag{1.2}$$

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*) The method described here for constructing solutions to the homogeneous equations is also applicable, with some modifications, to the nonaxisymmetric case.

provided that the function $m(z)$ satisfies the condition

$$d^2m(z) / dz^2 - \mu^2 m(z) = 0 \tag{1.3}$$

Here μ is a parameter which will be determined in satisfying the boundary conditions on the cylindrical boundary Γ_2 .

Substituting Equations (1.2) into (1.1) and taking into account (1.3), we obtain

$$m(z)\left[b'' + \frac{1}{r}b' + \frac{2(1-\nu)}{1-2\nu}b\mu^2 + \frac{1}{1-2\nu}a'\mu^2 + \frac{1}{1-2\nu}\frac{1}{r}a\mu^2\right] = 0$$

$$m'(z)\left[a'' + \frac{1}{r}a' + \frac{1-2\nu}{2(1-\nu)}a\mu^2 - \frac{1}{r^2}a + \frac{1}{2(1-\nu)}b'\right] = 0 \tag{1.4}$$

It is readily seen that the general solution of Equations (1.4) is given by

$$a(r) = A_1\mu^{-1}J_1 - A_2 rJ_0 + A_3\mu^{-1}Y_1 - A_4 rY_0$$

$$b(r) = -A_1J_0 + A_2[4(1-\nu)J_0 - \xi J_1] - A_3Y_0 + A_4[4(1-\nu)Y_0 - \xi Y_1] \tag{1.5}$$

Here $J_1 = J_1(\xi), J_0 = J_0(\xi)$ and $Y_1 = Y_1(\xi)$, $Y_0 = Y_0(\xi)$, are Bessel functions; $\xi = \mu r$; while A_i (i = 1, 2, 3, 4) are constants. In order that the solution (1.5) be defined for $\mu = 0$ as well, set $A_3 = A_3{}^*\mu^2$. Knowing $a(r)$, $b(r)$ and $m(z)$, we can find the displacements u and w as well as the stresses σ_z, σ_r, σ_θ and τ_{rz}; thus

$$u = m'(z)\frac{1}{\mu}[A_1J_1 - A_2\xi J_0 + A_3Y_1 - A_4\xi Y_0] \tag{1.6}$$

$$w = m(z)\{-A_1J_0 + A_2[4(1-\nu)J_0 - \xi J_1] - A_3Y_0 + A_4[4(1-\nu)Y_0 - \xi Y_1]\}$$

$$\sigma_z = \frac{E}{1+\nu}m'(z)\{-A_1J_0 + A_2[2(2-\nu)J_0 - \xi J_1] - A_3Y_0 + A_4[2(2-\nu)Y_0 - \xi Y_1]\}$$

$$\sigma_r = \frac{E}{1+\nu}m'(z)\left\{A_1\left(J_0 - \frac{1}{\xi}J_1\right) + A_2[\xi J_1 - (1-2\nu)J_0] + A_3\left(Y_0 - \frac{1}{\xi}Y_1\right) + A_4[\xi Y_1 - (1-2\nu)Y_0]\right\}$$

$$\sigma_\theta = \frac{E}{1+\nu}m'(z)\left[A_1\frac{1}{\xi}J_1 + A_2(2\nu-1)J_0 + A_3\frac{1}{\xi}Y_1 + A_4(2\nu-1)Y_0\right] \tag{1.7}$$

$$\tau_{rz} = \frac{E}{1+\nu}m(z)\mu\{A_1J_1 + A_2[2(\nu-1)J_1 - \xi J_0] + A_3Y_1 + A_4[2(\nu-1)Y_1 - \xi Y_0]\}$$

The constants A_1 through A_4 are found from the boundary conditions

$$\sigma_r(R_1, z) = 0, \quad \tau_{rz}(R_1, z) = 0, \quad \sigma_r(R_2, z) = 0, \quad \tau_{rz}(R_2, z) = 0 \quad \text{on } \Gamma_2 \tag{1.8}$$

Substituting the expressions for σ_r and τ_{rz} from (1.7) into (1.8), we obtain a system of linear algebraic equations in A_1, A_2, $A_3{}^*$ and A_4 . This system will have a nontrivial solution if the determinant of the coefficients vanishes.

This results in the following characteristic equation in μ :

$$\Delta(\mu) = \mu^2 \{[\xi_1^2 + 2(\nu - 1)][\xi_2^2 + 2(\nu - 1)] L_{11}^2 + \xi_1^2\xi_2^2 L_{00}^2 + [\xi_1^2 + 2(\nu - 1)]\xi_2^2 L_{10}^2 + [\xi_2^2 + 2(\nu - 1)]\xi_1^2 L_{01}^2 - 4(\nu - 1) - \xi_1^2 - \xi_2^2\} = 0 \qquad (1.9)$$

$$(L_{jk} = J_j(\xi_1) Y_k(\xi_2) - J_k(\xi_2) Y_j(\xi_1),\ \xi_1 = \mu R_1,\ \xi_2 = \mu R_2)$$

The transcendental equation (1.9) determines a countable set of parameters μ_k , together with the corresponding constants A_{1k}, A_{2k}, A_{3k} and A_{4k} the algebraic interrelationship among which is thus dependent on the characteristic determinant.

The first set of constants may then be written as

$$A_1 = \{[\xi_2 + 2(\nu - 1)\xi_2^{-1}] Y_1(\xi_2)[\xi_1 L_{01} - 2(\nu - 1) L_{11}] + \xi_2 Y_0(\xi_2)[\xi_1 L_{00} - 2(\nu - 1) L_{10}] + 2(\nu - 1)\xi_1\xi_2^{-1} Y_0(\xi_1) + [\xi_2 + 2(\nu - 1)\xi_2^{-1}] Y_1(\xi_1)\}\Omega$$

$$A_2 = \{[\xi_2 + 2(\nu - 1)\xi_2^{-1}] Y_1(\xi_2) L_{11} + \xi_2 Y_0(\xi_2) L_{10} - \xi_1\xi_2^{-1} Y_0(\xi_1)\}\Omega \qquad (1.10)$$

$$A_3 = -\{[\xi_2 + 2(\nu - 1)\xi_2^{-1}] J_1(\xi_2)[\xi_1 L_{01} - 2(\nu - 1) L_{11}] + \xi_2 J_0(\xi_2)[\xi_1 L_{00} - 2(\nu - 1) L_{10}] + 2(\nu - 1)\xi_1\xi_2^{-1} J_0(\xi_1) + [\xi_2 + 2(\nu - 1)\xi_2^{-1}] J_1(\xi_1)\}\Omega$$

$$A_4 = -\{[\xi_2 + 2(\nu - 1)\xi_2^{-1}] J_1(\xi_2) L_{11} + \xi_2 J_0(\xi_2) L_{10} - \xi_1\xi_2^{-1} J_0(\xi_1)\}\Omega$$

Here, Ω is a certain normalizing factor, and the index k has been omitted.

2. Analysis of the roots of the characteristic equation. Let us examine the behavior of the roots of Equation (1.9) when $R_1 \to R_2$. For convenience, we introduce a new parameter $\gamma = \mu R_1$ and set $\varepsilon = (R_2 - R_1)/R_1$, whereupon Equation (1.9) takes the form

$$\gamma^2 R_1^{-2}\Theta(\gamma, \varepsilon) = \gamma^2 R_1^{-2}\{[\gamma^2 + 2(\nu - 1)][\gamma^2(1 + \varepsilon)^2 + 2(\nu - 1)] L_{11}^2 + \gamma^4(1 + \varepsilon)^2 L_{00}^2 + [\gamma^2 + 2(\nu - 1)]\gamma^2(1 + \varepsilon)^2 L_{10}^2 + [\gamma^2(1 + \varepsilon)^2 + 2(\nu - 1)]\gamma^2 L_{01}^2 - 4(\nu - 1) - \gamma^2 - \gamma^2(1 + \varepsilon)^2\} = 0 \quad (2.1)$$

It is immediately clear that $\gamma_0 = 0$ is a double root of Equation (2.1). We will now prove that all remaining roots $\gamma_k \to \infty$ $(k = 1, 2, \ldots)$ when $\varepsilon \to 0$. The proof is obtained by contradiction. Assume initially that $\gamma_k \to \gamma_k^* \neq \infty$ when $\varepsilon \to 0$. Then, in the limit $\Theta(\gamma_k, \varepsilon) \to \varepsilon^2\Theta_0(\gamma_k^*)$, where $\Theta_0(\gamma_k^*)$ is independent of ε . Thus, the limit values of the set of roots γ_k as $\varepsilon \to 0$ are defined by Equation $\Theta_0(\gamma_k^*) = 0$. In the case under consideration, $\Theta_0(\gamma_k^*) \equiv 4(\nu^2 - 1)$, so that the assumption with regard to the existence of bounded roots is untenable.

Let us define more precisely the way in which the roots γ_k go to ∞ as $\varepsilon \to 0$. Let $\sigma_k = \varepsilon\gamma_k$. Then, in principle, as $\varepsilon \to 0$ the following limit relations are possible: 1) $\alpha_k \to 0$, 2) $\alpha_k \to$ const, 3) $\alpha_k \to \infty$. As previously shown, $\gamma_k \to \infty$ when $\varepsilon \to 0$, so that, by utilizing the asymptotic expressions for the Bessel functions, Equation (2.1) may be written as

$$\frac{\gamma^2}{R_1 R_2}\left\{[\sin^2\alpha_k - \alpha_k^2] + \varepsilon^2\left[\frac{8\nu-7}{4} + \frac{8\nu-7}{8}\frac{\sin 2\alpha_k}{\alpha_k} + \frac{8\nu^2-8\nu-1}{2}\frac{\sin^2\alpha_k}{\alpha_k^2}\right] + \right.$$
$$\left. + \varepsilon^3\left[-\frac{8\nu-7}{4} - \frac{8\nu-7}{8}\frac{\sin 2\alpha_k}{\alpha_k} - \frac{8\nu^2-8\nu-1}{2}\frac{\sin^2\alpha_k}{\alpha_k^2}\right] + \cdots\right\} = 0 \qquad (2.2)$$

In the first case mentioned above, $\alpha_k \to 0$ when $\varepsilon \to 0$. Making use of this property of small α_k and ε, Equation (2.2) may be written as

$$\gamma^2 R_1^{-1} R_2^{-1}\{[-{}^1/_3\,\alpha_k^4 + {}^2/_{45}\,\alpha_k^6 + \cdots] + \varepsilon^2[4(\nu^2-1) - {}^4/_3\,\alpha_k^2(\nu^2-1) +$$
$$+ {}^1/_{90}\,\alpha_k^4(16\nu^2+8\nu-23) + \cdots] + \varepsilon^3[-4(\nu^2-1) + {}^4/_3\,\alpha_k^2(\nu^2-1) -$$
$$- {}^1/_{90}\,\alpha_k^4(16\nu^2+8\nu-23) + \cdots] + \cdots\} = 0 \qquad (2.3)$$

From Equation (2.3), we obtain the asymptotic expansion

$$\gamma_k = \frac{\delta_k}{\sqrt{\varepsilon}}, \qquad \delta_k = \gamma_{0k} + \varepsilon\gamma_{1k} + \varepsilon^2\gamma_{2k} + \cdots, \qquad \gamma_{0k}^4 - 12(\nu^2-1) = 0 \quad (2.4)$$

$$\gamma_{1k} = \frac{3}{5}(1-\nu^2)\frac{1}{\gamma_{0k}} - \frac{1}{4}\gamma_{0k}$$
$$\gamma_{2k} = \left(\frac{229}{2100} + \frac{1}{15}\nu + \frac{863}{16800}\nu^2\right)\gamma_{0k} + \frac{9}{20}(\nu^2-1)\frac{1}{\gamma_{0k}} \qquad (2.5)$$

Now let us examine the second case, $\alpha_k \to \alpha_{0k}$ when $\varepsilon \to 0$. In this case, it is readily seen from (2.2) that α_{0k} satisfies the Equation

$$\frac{1}{\alpha_{0k}^4}(\sin^2\alpha_{0k} - \alpha_{0k}^2) = 0 \qquad (2.6)$$

It is important to note that Equation (2.6) actually coincides with the equation defining the exponents associated with the edge effects in the theory of plates given by St.Veneant [2 and 3]. Since Equation (2.6) has a countable set of roots, Equation (2.2) also has a countable set of roots such that $\gamma_k\varepsilon \to$ const, when $\varepsilon \to 0$. A more precise evaluation of the roots under consideration may be obtained by means of the expansion

$$\gamma_k = \frac{\Delta_k}{\varepsilon}; \quad \Delta_k = \delta_{0k} + \varepsilon^2\delta_{2k} + \varepsilon^3\delta_{3k} + \cdots, \quad \frac{1}{\delta_{0k}^4}(\sin^2\delta_{0k} - \delta_{0k}^2) = 0 \quad (2.7)$$

$$\delta_{2k} = \frac{2(1-\nu^2)}{\sin 2\delta_{0k} - 2\delta_{0k}} + \frac{7-8\nu}{16\delta_{0k}}, \qquad \delta_{3k} = -\delta_{2k} \qquad (2.8)$$

We will show that the third case cannot exist. Indeed, from (2.2) it is clear that if $\varepsilon \to 0$, it is impossible to satisfy the asymptotic relations $\sin^2\alpha_k \sim \alpha_k^2$ for α_k continuously tending to infinity.

The preceding analysis shows that the characteristic equation (2.1) has three groops of roots:

1) The double root $\gamma_0 = 0$, which is independent of ε;

2) Four roots γ_k which are defined by Formulas (2.4) and (2.5) and which increase like $1/\sqrt{\varepsilon}$ as $\varepsilon \to 0$;

3) A countable set of roots defined by Equations (2.7) and (2.8) and

increasing like $\frac{1}{\epsilon}$ as $\epsilon \to 0$.

3. Analysis of the state of stress and deformation corresponding to each group of roots. Group (1) . Corresponding to the double root $\gamma_0 = 0$, we have

$$m_0(z) = \frac{A_0}{1+\nu} z, \qquad a(r) = -\nu r, \qquad b(r) = 1 \tag{3.1}$$

The displacements and stresses are given by

$$u = -R_1 \frac{\nu}{1+\nu} A_0 \rho, \quad w = R_1 \frac{1}{1+\nu} A_0 \zeta \quad \left(\rho = \frac{r}{R_1}, \quad \zeta = \frac{z}{R_1}\right) \tag{3.2}$$

$$\sigma_z = 2GA_0, \quad \sigma_r = 0, \quad \sigma_\theta = 0, \quad \tau_{rz} = 0 \tag{3.3}$$

Here, ρ and ζ are nondimensional coordinates, and G is the shear modulus. Thus, the first group of roots $\gamma_0 = 0$ corresponds to pure extension in the direction of the axis of symmetry. This state of stress is propagated without attenuation into the interior region of the shell.

Group (2) . The function $m_k(z)$ is obtained from Equation

$$m_k'' - \gamma_k^2 / R_1^2 m_k = 0 \qquad (\gamma_k = \delta_k / \sqrt{\varepsilon})$$

where δ_k is as given in (2.4). Whence,

$$m_k(z) = R_1 \left(E_k \exp \frac{\delta_k \zeta}{\sqrt{\varepsilon}} + N_k \exp \frac{-\delta_k \zeta}{\sqrt{\varepsilon}}\right) \tag{3.4}$$

where E_k and N_k are constants of integration which are determined from the boundary conditions on the end faces Γ_1 .

$$u(r, z) = u_1(\gamma_1 \rho, \gamma_1 \zeta) + u_2(\gamma_2 \rho, \gamma_2 \zeta) \tag{3.5}$$
$$w(r, z) = w_1(\gamma_1 \rho, \gamma_1 \zeta) + w_2(\gamma_2 \rho, \gamma_2 \zeta)$$

$$\begin{aligned}
\sigma_z(r, z) &= \sigma_{z1}(\gamma_1 \rho, \gamma_1 \zeta) + \sigma_{z2}(\gamma_2 \rho, \gamma_2 \zeta) \\
\sigma_r(r, z) &= \sigma_{r1}(\gamma_1 \rho, \gamma_1 \zeta) + \sigma_{r2}(\gamma_2 \rho, \gamma_2 \zeta) \\
\sigma_\theta(r, z) &= \sigma_{\theta 1}(\gamma_1 \rho, \gamma_1 \zeta) + \sigma_{\theta 2}(\gamma_2 \rho, \gamma_2 \zeta) \\
\tau_{rz}(r, z) &= \tau_{rz1}(\gamma_1 \rho, \gamma_1 \zeta) + \tau_{rz2}(\gamma_2 \rho, \gamma_2 \zeta)
\end{aligned} \tag{3.6}$$

In Expressions (3.5) and (3.6), the quantities $u_k, w_k, \sigma_{zk}, \sigma_{rk}, \sigma_{\theta k}$ and τ_{rzk}, i.e. the displacements and stresses corresponding to the root of the second group γ_k , are obtained from (1.6), (1.7), (1.10) and (3.4) upon setting

$$\mu = \gamma_k / R_1, \quad \xi = \gamma_k \rho, \quad \xi_1 = \gamma_k, \quad \xi_2 = \gamma_k(1+\varepsilon), \quad \Omega = \gamma_k^2 / \sqrt{\varepsilon}$$

The summation is carried out over those roots γ_k for which $\operatorname{Re}\{\gamma_k\} > 0$. Expanding the solutions for the second group for small values of ϵ , we obtain the following asymptotic expressions:

$$m_k(z) = R_1 \left[m_k^* + \sqrt{\varepsilon} \zeta \gamma_{1k} \frac{dm_k^*}{d\eta_k} + \varepsilon \frac{\zeta^2}{2} \gamma_{1k}^2 m_k^* + \right. \tag{3.7}$$
$$\left. + \varepsilon \sqrt{\varepsilon} \left(\frac{\zeta^3}{6} \gamma_{1k}^3 + \zeta \gamma_{2k} \right) \frac{dm_k^*}{d\eta_k} + \cdots \right]$$

$$m_k'(z) = \frac{\gamma_{0k}}{\sqrt{\varepsilon}}\left[\frac{dm_k^*}{dr_k} + \sqrt{\varepsilon}\zeta\gamma_{1k}m_k^* + \varepsilon\left(\frac{\gamma_{1k}}{\gamma_{0k}} + \frac{\zeta^2}{2}\gamma_{1k}^2\right)\frac{dm_k^*}{dr_k} + \right. \qquad (3.7)\ \text{cont.}$$

$$\left. + \varepsilon\sqrt{\varepsilon}\left(\frac{\zeta^3}{6}\gamma_{1k}^3 + \zeta\gamma_{2k} + \zeta\frac{\gamma_{1k}^2}{\gamma_{0k}}\right)m_k^* + \cdots\right]$$

$$\left(m_k^* = E_k e^{\eta_k} + N_k e^{-\eta_k}, \quad r_k = \frac{\gamma_{0k}}{\sqrt{\varepsilon}}\zeta\right)$$

$$u_k = m_k'(z)R_1\{\sqrt{\varepsilon}[-4(\nu-1)] + \varepsilon\sqrt{\varepsilon}[2(\nu-1)(3-\nu) + 2\nu(\nu-1)r_0 - {}^4/_3\gamma_{0k}^2 + \nu({}^5/_6 + {}^1/_2 r_0^2)\gamma_{0k}^2] + \cdots\} \qquad (3.8)$$

$$w_k = m_k(z)\{\sqrt{\varepsilon}[4(\nu-1)\nu + 2\gamma_{0k}^2(\nu-1)r_0] + \varepsilon\sqrt{\varepsilon}[2\nu(\nu-4) + (\nu+1)(r_0^3 - 3r_0) - (\nu^2-1)(r_0^3 + {}^{37}/_5 r_0) + (\nu+1)\gamma_{0k}^2(-{}^7/_6 + r_0 - {}^1/_2 r_0^2) - \gamma_{0k}^2({}^3/_2 + 3r_0 + {}^1/_2 r_0^2)](\nu-1) + \cdots\} \qquad (3.9)$$

$$\tau_{rzk} = 2Gm_k(z)R_1^{-1}\{\sqrt{\varepsilon}[6(\nu^2-1)(r_0^2-1)] + \varepsilon\sqrt{\varepsilon}[(\nu-1)(r_0^2-1)(1-{}^1/_3 r_0) + (1-r_0^2)(4+{}^2/_3 r_0) + \gamma_{0k}^2(1-r_0^2)({}^{13}/_{20} + {}^1/_{12} r_0^2)]3(\nu^2-1) + \cdots\} \qquad (3.10)$$

$$\sigma_{zk} = 2Gm_k'(z)\{\sqrt{\varepsilon}[-2\gamma_{0k}^2 r_0] + \varepsilon\sqrt{\varepsilon}[(\nu^2-1)({}^{22}/_5 r_0 + 2r_0^3) - (\nu-1)({}^1/_6 + r_0 - {}^1/_2 r_0^2)\gamma_{0k}^2 + ({}^1/_6 + 3r_0 + {}^1/_2 r_0^2)\gamma_{0k}^2] + \cdots\} \qquad (3.11)$$

$$\sigma_{\theta k} = 2Gm_k'(z)\{\sqrt{\varepsilon}[-4(\nu^2-1) - 2\nu\gamma_{0k}^2 r_0] + \varepsilon\sqrt{\varepsilon}[2(\nu^2-1)(4-\nu) + 2(\nu^2-1)(1+{}^{37}/_{10}\nu)r_0 + \nu(\nu^2-1)r_0^3 - {}^4/_3\gamma_{0k}^2 + {}^1/_6\nu(7\nu-1)\gamma_{0k}^2 + \nu(4-\nu)\gamma_{0k}^2 r_0 + {}^1/_2\nu(1+\nu)\gamma_{0k}^2 r_0^2] + \cdots\} \qquad (3.12)$$

$$\sigma_{rk} = 2Gm_k'(z)\{\varepsilon\sqrt{\varepsilon}[(\nu^2-1)(r_0 - r_0^3) + {}^1/_2\nu\gamma_{0k}^2(1-r_0^2)] + \cdots\} \qquad (3.13)$$

Here, the new coordinate r_0 is measured from the middle surface. Its relationship to ρ is given by

$$\rho = 1 + {}^1/_2\varepsilon(1 + r_0), \qquad -1 \leqslant r_0 \leqslant 1 \qquad (3.14)$$

From Expressions (3.7) to (3.13), it can be seen that, when ε is small, u_k, σ_{zk} and $\sigma_{\theta k}$ are of the order of unity; w_k and τ_{rzk} are of order $\sqrt{\varepsilon}$, while σ_{rk} is of order ε.

Thus, the solutions corresponding to the second group of roots represent edge effects which decrease towards the interior region of the shell like $\exp(-\delta_k n/\sqrt{\varepsilon})$, where n is the distance from the end face Γ_1 measured along the normal to the face.

To clarify the pattern of the stress distribution which corresponds to the group of roots under consideration, we will determine the stress resultant and moment resultant due to the stresses at a section ζ = const

$$P_k + iT_k = \int_{R_1}^{R_2}(\sigma_{zk} + i\tau_{rzk})r\,dr, \qquad M_k = \int_{R_1}^{R_2}\sigma_{zk}r^2 dr$$

from which we obtain

$$P_k = 0, \qquad T_k = \frac{m_k(z)}{m_k'(z)} \frac{\gamma_k^2}{R_1^2} M_k$$

$$M_k = 2Gm_k'(z)\, R_1^3 \{\varepsilon^2 \sqrt[2]{\varepsilon}\, [-{}^1/_3 \gamma_{0k}^2] + \qquad (3.15)$$

$$+ \varepsilon^3 \sqrt[3]{\varepsilon}\, [{}^{14}/_{15} (\nu^2 - 1) + {}^1/_6 \gamma_{0k}^2 \; (3 - \nu)] + \cdots\} \neq 0$$

Thus, T_k and M_k are of order $\epsilon\sqrt{\epsilon}$, and ϵ^2, respectively. Hence it is possible, with the aid of the foregoing computations, to remove the stress resultant and moment resultant due to a given system of stresses by appropriately loading the end faces, i.e. we can obtain

$$\int_{R_1}^{R_2} \tau_{rz} r dr = 0, \qquad \int_{R_1}^{R_2} \sigma_z r^2 dr = 0$$

Group (3). The finction $m_p(z)$ must be such that

$$m_p'' - \gamma_p^2 / R_1^2 m_p = 0 \qquad (\gamma_p = \varepsilon^{-1} \Delta p)$$

where Δ_p is as given in (2.7) and (2.8). Thus,

$$m_p(z) = R_1 [E_p^* \exp(\varepsilon^{-1} \zeta \Delta p) + N_p^* \exp(-\varepsilon^{-1} \zeta \Delta \rho)] \qquad (3.16)$$

The displacements and stresses are obtained here by means of Formulas (1.6), (1.7), (1.10) and (3.16) in which γ_p is the corresponding root in the third group for which $\mathrm{Re}\{\gamma_p\} > 0$ and $\Omega = \gamma_p$. The states of stress corresponding to the third group of roots represent edge effects which decrease towards the interior of the shell like $\exp(-\epsilon^{-1} n \Delta_p)$. Expanding the solutions of this group in powers of the small parameter ϵ, we obtain the following asymptotic axpressions:

$$m_p(z) = R_1 \Big[m_p^* + \varepsilon \delta_{2p} \zeta \frac{dm_p^*}{d\lambda_p} + \varepsilon^2 \Big(m_p^* \frac{\zeta^2}{2} \delta_{2p}^2 + \delta_{3p} \zeta \frac{dm_p^*}{d\lambda_p} \Big) + \cdots \Big]$$

$$m_p'(z) = \frac{\delta_{0p}}{\varepsilon} \Big[\frac{dm_p^*}{d\lambda_p} + \varepsilon \delta_{2p} \zeta m_p^* + \qquad (3.17)$$

$$+ \varepsilon^2 \Big(\frac{dm_p^*}{d\lambda_p} \frac{\zeta^2}{2} \delta_{2p}^2 + \frac{dm_p^*}{d\lambda_p} \frac{\delta_{2p}}{\delta_{0p}} + m_p^* \delta_{3p} \zeta \Big) + \cdots \Big]$$

$$\Big(m_p^* = E_p^* e^{\lambda_p} + N_p^* e^{-\lambda_p}, \quad \lambda_p = \frac{\delta_{0p}}{\varepsilon} \zeta \Big)$$

$$u_p = m_p'(z) R_1 (u_{0p} \varepsilon^2 + u_{1p} \varepsilon^3 + \cdots), \qquad w_p = m_p(z) (w_{0p} \varepsilon + w_{1p} \varepsilon^2 + \cdots)$$

$$\sigma_{zp} = 2Gm_p'(z) (\sigma_{z0p} \varepsilon + \sigma_{z1p} \varepsilon^2 + \cdots)$$

$$\tau_{rzp} = 2Gm_p(z) R_1^{-1} (\tau_{rz0p} + \tau_{rz1p} \varepsilon + \cdots) \qquad (3.18)$$

$$\sigma_{\theta p} = 2Gm_p'(z) (\sigma_{\theta 0p} \varepsilon + \sigma_{\theta 1p} \varepsilon^2 + \cdots), \qquad \sigma_{rp} = 2Gm_p'(z) (\sigma_{r0p} \varepsilon + \sigma_{r1p} \varepsilon^2 + \cdots)$$

$$u_{0p} = \left[\frac{\sin 2\delta_{0p}}{2\delta_{0p}} r_1 + r_1 - 2\nu + 1\right] \sin \delta_{0p} r_1 + \left[(1-\nu)\left(\frac{\sin 2\delta_{0p}}{\delta_{0p}^2} + \frac{2}{\delta_{0p}}\right) - \delta_{0p} r_1\right] \cos \delta_{0p} r_1 \quad \text{(cont.)}$$

$$u_{1p} = \left[-\frac{\sin 2\delta_{0p}}{2\delta_{0p}}\left(r_1 + \frac{r_1^2}{2}\right) + 1 - \frac{1}{\delta_{0p}^2} + \frac{r_1}{2} - \frac{r_1^2}{2} + (\nu - 1)\left(\frac{\sin 2\delta_{0p}}{\delta_{0p}^3} + 2 - \frac{2}{\delta_{0p}^2} + 3r_1\right) + 2(\nu-1)^2 \frac{\sin 2\delta_{0p}}{\delta_{0p}^3}\right] \sin \delta_{0p} r_1 + \left[\frac{\sin 2\delta_{0p}}{2\delta_{0p}^2} r_1 + \delta_{0p} r_1 + \frac{r_1^2}{2}\delta_{0p} + (\nu - 1)\left(\frac{\sin 2\delta_{0p}}{\delta_{0p}^2} + \frac{3}{2}\frac{\sin 2\delta_{0p}}{\delta_{0p}^2} r_1 + \frac{r_1}{\delta_{0p}}\right) - \frac{4}{\delta_{0p}}(\nu - 1)^2\right] \cos \delta_{0p} r_1$$

$$w_{0p} = \left[\frac{\sin 2\delta_{0p}}{2\delta_{0p}} + 1 - r_1 \delta_{0p}^2 + (\nu - 1)\left(\frac{\sin 2\delta_{0p}}{\delta_{0p}} + 2\right)\right] \sin \delta_{0p} r_1 + \left[-\frac{r_1}{2} \sin 2\delta_{0p} - r_1 \delta_{0p} - 2(\nu - 1)\delta_{0p}\right] \cos \delta_{0p} r_1$$

$$w_{1p} = \left[-\frac{\sin 2\delta_{0p}}{2\delta_{0p}} - \frac{1}{2} - r_1 + \delta_{0p}^2\left(r_1 + \frac{r_1^2}{2}\right) + (\nu - 1)\left(\frac{\sin 2\delta_{0p}}{2\delta_{0p}} r_1 - \frac{\sin 2\delta_{0p}}{\delta_{0p}} + 1 - r_1\right)\right] \sin \delta_{0p} r_1 + \left[\sin 2\delta_{0p}\left(\frac{r_1}{2} + \frac{r_1^2}{4}\right) + \delta_{0p}\left(\frac{r_1}{2} + \frac{r_1^2}{2}\right) + (\nu - 1)\left(\frac{3}{2}\frac{\sin 2\delta_{0p}}{\delta_{0p}^2} + \frac{1}{\delta_{0p}} + 2\delta_{0p} - r_1\delta_{0p}\right)\right] \cos \delta_{0p} r_1$$

$$\sigma_{z0p} = \left[-\frac{\sin 2\delta_{0p}}{2\delta_{0p}} - 1 - r_1 \delta_{0p}^2\right] \sin \delta_{0p} r_1 + \left[-\frac{r_1}{2} \sin 2\delta_{0p} + 2\delta_{0p} - r_1 \delta_{0p}\right] \cos \delta_{0p} r_1$$

$$\sigma_{z1p} = \left[\frac{\sin 2\delta_{0p}}{2\delta_{0p}}(1 + r_1) + \frac{1}{2} + \delta_{0p}^2\left(r_1 + \frac{r_1^2}{2}\right) + (\nu - 1)\left(\frac{\sin 2\delta_{0p}}{\delta_{0p}} r_1 - 2\right)\right] \sin \delta_{0p} r_1 + \left[-\frac{\sin 2\delta_{0p}}{2\delta_{0p}^2} + \sin 2\delta_{0p}\left(\frac{r_1}{2} + \frac{r_1^2}{4}\right) - 2\delta_{0p} + \frac{1}{\delta_{0p}} + \delta_{0p}\left(\frac{r_1^2}{2} - \frac{r_1}{2}\right) + (\nu - 1)\left(-\frac{\sin 2\delta_{0p}}{\delta_{0p}^2} + \frac{2}{\delta_{0p}} - 2r_1\delta_{0p}\right)\right] \cos \delta_{0p} r_1$$

$$\tau_{rz0p} = \left[\frac{r_1}{2} \delta_{0p} \sin 2\delta_{0p} - \delta_{0p}^2 (1 - r_1)\right] \sin \delta_{0p} r_1 - r_1 \delta_{0p}^3 \cos \delta_{0p} r_1$$

$$\tau_{rz1p} = \left[-\delta_{0p} \sin 2\delta_{0p}\left(\frac{r_1}{2} + \frac{r_1^2}{4}\right) - 1 + \delta_{0p}^2\left(1 + \frac{r_1}{2} - \frac{r_1^2}{2}\right) + (\nu - 1)(2r_1\delta_{0p}^2 - 2)\right] \sin \delta_{0p} r_1 + \left[\frac{r_1}{2} \sin 2\delta_{0p} + \delta_{0p}^3\left(r_1 + \frac{r_1^2}{2}\right) + (\nu - 1) r_1 \sin 2\delta_{0p}\right] \cos \delta_{0p} r_1$$

(cont.)

$$\sigma_{r0p} = \left[-\frac{\sin 2\delta_{0p}}{2\delta_{0p}} + \delta_{0p}^2 r_1 - 1\right]\sin\delta_{0p}r_1 + \left[\frac{r_1}{2}\sin 2\delta_{0p} + r_1\delta_{0p}\right]\cos\delta_{0p}r_1$$

$$\sigma_{r1p} = \left[-\frac{\sin 2\delta_{0p}}{2\delta_{0p}}(r_1-1) + \frac{3}{2} - \delta_{0p}^2\left(r_1 + \frac{r_1^2}{2}\right) - (\nu-1)\frac{\sin 2\delta_{0p}}{\delta_{0p}}\ r_1\right]\sin\delta_{0p}r_1 + \left[-\sin 2\delta_{0p}\left(\frac{r_1}{2}+\frac{r_1^2}{4}\right) + \delta_{0p}\left(\frac{r_1}{2}-\frac{r_1^2}{2}\right) + 2r_1(\nu-1)\,\delta_{0p}\right]\cos\delta_{0p}r_1$$

$$\sigma_{\theta 0p} = \left(-\nu\frac{\sin 2\delta_{0p}}{\delta_{0p}} - 2\nu\right)\sin\delta_{0p}r_1 + 2\nu\,\delta_{0p}\cos\delta_{0p}r_1$$

$$\sigma_{\theta 1p} = \left[\frac{\sin 2\delta_{0p}}{\delta_{0p}}(1+r_1) + 2r_1 + (\nu-1)\left(\frac{\sin 2\delta_{0p}}{\delta_{0p}} + \frac{r_1}{2}\frac{\sin 2\delta_{0p}}{\delta_{0p}} - 5 + r_1\right) - 4(\nu-1)^2\right]\sin\delta_{0p}r_1 + \left[-\frac{\sin 2\delta_{0p}}{2\delta_{0p}^2} - 2\delta_{0p}(1+r_1) + \frac{1}{\delta_{0p}} - 2(\nu-1)^2\frac{\sin 2\delta_{0p}}{\delta_{0p}^2} + (\nu-1)\left(-\frac{7}{2}\frac{\sin 2\delta_{0p}}{\delta_{0p}^2} - \frac{1}{\delta_{0p}} - 2\delta_{0p} - \delta_{0p}r_1\right)\right]\cos\delta_{0p}r_1 \qquad (3.19)$$

Here r_1 is a new coordinate, measured from the inner cylindrical surface

$$\rho = 1 + \varepsilon r_1 \qquad (0 \leqslant r_1 \leqslant 1) \qquad (3.20)$$

From (3.17) to (3.19) we note that the displacements u_p and w_p are of order ε and the stresses $\sigma_{zp}, \tau_{rzp}, \sigma_{\theta p}$ and σ_{rp} are of the order of unity.

If we refer back to the coordinate r_0 , measured from the middle surface, u_{0p} and τ_{rz0p} will be even functions while w_{0p}, σ_{z0p}, $\sigma_{\theta 0p}$ and σ_{r0p} will be odd functions of r_0 when

$$(A) \qquad \delta_{0p} = \sin\delta_{0p} \qquad (3.21)$$

On the other hand, u_{0p} and τ_{rz0p} will be odd functions while w_{0p}, σ_{z0p}, $\sigma_{\theta 0p}$ and σ_{r0p} will be even functions of r_0 when

$$(B) \qquad \delta_{0p} = -\sin\delta_{0p} \qquad (3.22)$$

From the above, we find that the roots $\sigma_p = 2(\sigma_{0p}/\varepsilon + \sigma_{2p}\varepsilon + \ldots)$, for which the relations $(\sin 2\sigma_{0p} - 2\sigma_{0p})/\sigma_{0p}^3 = 0$, hold, correspond to solutions representing primarily shell bending, whereas roots $\omega_p = 2(\omega_{0p}/\varepsilon + \omega_{2p}\varepsilon + \ldots)$, for which the relations

$$(\sin 2\omega_{0p} + 2\omega_{0p})/\omega_{0p} = 0,$$

hold, correspond to solutions representing primarily extensional deformations of the sell. Thus, we have

for group (A)

$$u_{0p} = -2(\nu - 1)\sigma_{0p}^{-1}\cos\sigma_{0p}\cos\sigma_{0p}r_0 - \sin\sigma_{0p}\cos\sigma_{0p}r_0 + r_0\sin\sigma_{0p}r_0\cos\sigma_{0p}$$
$$w_{0p} = 2\sin\sigma_{0p}r_0(\cos\sigma_{0p} - \sigma_{0p}\sin\sigma_{0p}) - 2r_0\sigma_{0p}\cos\sigma_{0p}r_0\cos\sigma_{0p} + 4(\nu - 1)\sin\sigma_{0p}r_0\cos\sigma_{0p}$$
$$\sigma_{z0p} = -2\sin\sigma_{0p}r_0\cos\sigma_{0p} - 2\sigma_{0p}\sin\sigma_{0p}r_0\sin\sigma_{0p} - 2r_0\sigma_{0p}\cos\sigma_{0p}r_0\cos\sigma_{0p} \quad (3.23)$$
$$\tau_{rz0p} = -4\sigma_{0p}^2\cos\sigma_{0p}r_0\sin\sigma_{0p} + 4\sigma_{0p}^2 r_0\sin\sigma_{0p}r_0\cos\sigma_{0p}$$
$$\sigma_{r0p} = 2r_0\sigma_{0p}\cos\sigma_{0p}r_0\cos\sigma_{0p} - 2\sin\sigma_{0p}r_0(\cos\sigma_{0p} - \sigma_{0p}\sin\sigma_{0p})$$
$$\sigma_{\theta 0p} = -4\nu\sin\sigma_{0p}r_0\cos\sigma_{0p}$$

for group (B) (3.24)

$$u_{0p} = 2(\nu - 1)\omega_{0p}^{-1}\sin\omega_{0p}\sin\omega_{0p}r_0 - \sin\omega_{0p}r_0\cos\omega_{0p} + r_0\cos\omega_{0p}r_0\sin\omega_{0p}$$
$$w_{0p} = 2\cos\omega_{0p}r_0(\sin\omega_{0p} + \omega_{0p}\cos\omega_{0p}) + 2r_0\omega_{0p}\sin\omega_{0p}r_0\sin\omega_{0p} + 4(\nu - 1)\cos\omega_{0p}r_0\sin\omega_{0p}$$
$$\sigma_{z0p} = -2\cos\omega_{0p}r_0\sin\omega_{0p} + 2\omega_{0p}\cos\omega_{0p}r_0\cos\omega_{0p} + 2r_0\omega_{0p}\sin\omega_{0p}r_0\sin\omega_{0p}$$
$$\tau_{rz0p} = -4\omega_{0p}^2\sin\omega_{0p}r_0\cos\omega_{0p} + 4\omega_{0p}^2 r_0\cos\omega_{0p}r_0\sin\omega_{0p}$$
$$\sigma_{r0p} = -2r_0\omega_{0p}\sin\omega_{0p}r_0\sin\omega_{0p} - 2\cos\omega_{0p}r_0(\sin\omega_{0p} + \omega_{0p}\cos\omega_{0p})$$
$$\sigma_{\theta 0p} = -4\nu\cos\omega_{0p}r_0\sin\omega_{0p}$$

Now let us examine the stress resultants and moment resultants at a section ζ = const . Thus (3.25)

$$P_p = \int_{R_1}^{R_2}\sigma_{zp}r\,dr = 0, \qquad M_p = \int_{R_1}^{R_2}\sigma_{zp}r^2 dr \neq 0, \qquad T_p = \frac{m_p(z)}{m_p'(z)}\frac{\gamma_p^2}{R_1^2}M_p$$

$$M_p = 2Gm_p'(z)R_1^3\Big\{\varepsilon^4\Big[-\frac{\sin 2\delta_{0p}}{\delta_{0p}^4} + \frac{2\cos\delta_{0p}}{\delta_{0p}^3} + \frac{2\sin\delta_{0p}}{\delta_{0p}^4} - \frac{2}{\delta_{0p}^3} + (\nu - 1)\Big(-\frac{\sin 2\delta_{0p}}{\delta_{0p}^4} + \frac{2\cos\delta_{0p}}{\delta_{0p}^3} + \frac{2\sin\delta_{0p}}{\delta_{0p}^4} - \frac{2}{\delta_{0p}^3}\Big)\Big] + \varepsilon^5\Big[\frac{2}{\delta_{0p}^3} + \frac{\sin 2\delta_{0p}}{\delta_{0p}^4} - \frac{3\cos\delta_{0p}}{\delta_{0p}^3} - \frac{\sin\delta_{0p}}{\delta_{0p}^4} + (\nu - 1)\Big(\frac{\sin 2\delta_{0p}}{\delta_{0p}^4} - \frac{3\cos\delta_{0p}}{\delta_{0p}^3} - \frac{3\sin\delta_{0p}}{\delta_{0p}^4} - \frac{4}{\delta_{0p}^3}\Big) - \frac{4}{\delta_{0p}^3}(\nu - 1)^2\Big] + \cdots\Big\}$$

It is apparent from (3.25) that T_p and M_p corresponding to the root σ_p are of order ε^3 and ε^4, respectively, whereas those corresponding to the root ω_p are of order ε^2 and ε^3, respectively.

Thus, a given system of stresses at a section ζ = const can be taken to vanish with accuracies of ε^2 and ε^3 for the stress resultant and moment resultant, respectively.

All of the foregoing provides a basis for the conclusion that the edge effects of applied shell theories correspond to the second group of solutions. The third group of solutions represents edge effects of the St. Venant type, which are completely absent from the Kirchhoff theory.

From the preceding investigations, we may draw some conclusions concerning the accuracy of applied shell theories.

1) The Vlasov theory [4]. For axisymmetric deformation we obtain the relations

$$\frac{\partial^2 w}{\partial \zeta^2} + \nu \frac{\partial u}{\partial \zeta} - c^2 \frac{\partial^3 u}{\partial \zeta^3} = \frac{1-\nu^2}{Eh} R^2 X \qquad \left(c^2 = \frac{h^2}{12R^2}, \quad R = 0.5\,(R_1 + R_2)\right) \quad (3.26)$$

$$\nu \frac{\partial w}{\partial \zeta} - c^2 \frac{\partial^3 w}{\partial \zeta^3} + c^2 \frac{\partial^4 u}{\partial \zeta^4} + (1 + c^2)\, u = -\frac{1-\nu^2}{Eh} R^2 Z$$

Here, u is the radial displacement of a point on the middle surface ; w is the displacement along the generator ; X and Z are the tangential and normal components of external loading.

The corresponding characteristic equation is given by

$$(1-\nu^2) + \varepsilon^2 ({}^1/_{12} + {}^1/_6 \nu\gamma^2 + {}^1/_{12}\gamma^4) + \varepsilon^3 (-{}^1/_{12} - {}^1/_6 \nu\gamma^2 - {}^1/_{12}\gamma^4) +$$
$$+ \varepsilon^4 ({}^1/_{16} + {}^1/_8 \nu\gamma^2 + {}^1/_{18}\gamma^4) + \ldots = 0 \qquad (3.27$$

From (3.27), we obtain an expansion of the exponent associated with the edge effect for the shell theory under consideration

$$\gamma_k = \frac{\delta_k}{\sqrt{\varepsilon}}, \qquad \delta_k = \gamma_{0k} + \varepsilon\gamma_{1k} + \ldots,$$

$$\gamma_{0k}{}^4 - 12\,(\nu^2 - 1) = 0, \qquad \gamma_{1k} = \frac{1}{4}\gamma_{0k} - \frac{\nu}{2}\frac{1}{\gamma_{0k}} \qquad (3.28)$$

2) The Darevskii theory [5]. The characteristic equation in this case is

$$12\,(1-\nu^2)\,\gamma^4 + \varepsilon^2\,[(4 - 3\nu^2)\,\gamma^4 + 2\nu\gamma^6 +$$
$$+ \gamma^8] + \varepsilon^3\,[(3\nu^2 - 4)\,\gamma^4 - 2\nu\gamma^6 - \gamma^8] + \ldots = 0 \qquad (3.29)$$

from which we obtain the expansion of the exponent associated with the edge effect

$$\gamma_k = \frac{\delta_k}{\sqrt{\varepsilon}}, \qquad \delta_k = \gamma_{0k} + \varepsilon\gamma_{1k} + \ldots,$$

$$\gamma_{0k}{}^4 - 12\,(\nu^2 - 1) = 0, \qquad \gamma_{1k} = \frac{1}{4}\gamma_{0k} - \frac{\nu}{2}\frac{1}{\gamma_{0k}} \qquad (3.30)$$

Comparing Equations (3.28) and (3.30) with the exact expansion (2.4) and (2.5), we find that the first terms coincide, but subsequent terms differ. The same conclusion is obtained for all other cylindrical shell theories.

Thus, an analysis of existing shell theories shows that they approximate the second group type of stress with first order accuracy, but none of them can make any claim to second order accuracy, since in none of these theories does the second order term coincide with the exact value given in Formulas (2.4) and (2.5).

4. Satisfaction of the boundary conditions on the end faces of the cylinder using the solutions to the homogeneous equations. We will now study in detail the problem of balancing the system of stresses on the end faces Γ_1. Assume that the stresses on the end faces $\zeta_1 = \frac{l}{R_1}$ and $\zeta_2 = -\frac{l}{R_2}$ are given by σ_{z1}, τ_{rz1} and σ_{z2}, τ_{rz2}, respectively. Whereupon it is sufficient to consider the following cases :

1) The loading is symmetric about the plane $\zeta = 0$

$$\sigma_{z1} = \sigma_{z2}, \qquad \tau_{rz1} = -\tau_{rz2}$$

2) The loading is antisymmetric about the plane $\zeta = 0$

$$\sigma_{z1} = -\sigma_{z2}, \qquad \tau_{rz1} = \tau_{rz2}$$

In the first case, we can set $m_0 = \zeta$, $m_k = \sinh \gamma_k \zeta$; in the second case, we take $m_k = \cosh \gamma_k \zeta$. The development will be confined to the first case, since the results with regard to the second case can be obtained from the first case by replacing $\sinh \gamma_k \zeta$ with $\cosh \gamma_k \zeta$. As a preliminary step, we will obtain the solution corresponding to pure extension in the ζ direction

$$u = -\frac{\nu}{1+\nu} R_1 A_0 \rho, \quad w = R_1 \frac{A_0}{1+\nu}\zeta, \ \sigma_z = 2GA_0, \quad \sigma_r = \sigma_\theta = \tau_{rz} = 0 \quad (4.1)$$

$$A_0 = \frac{1}{2GS_0}\int_{R_1}^{R_2} \sigma_{z1} r dr, \qquad S_0 = \int_{R_1}^{R_2} r dr$$

The remaining self-equilibrating system of normal stresses will be designated as σ_{z1}^*

$$\sigma_{z1}^* = \sigma_{z1} - \frac{1}{S_0}\int_{R_1}^{R_2} \sigma_{z1} r\, dr \qquad (4.2)$$

We seek a general solution to this problem of the form

$$u = \sum_{k=1}^{2} u_k(\gamma_k\rho, \gamma_k\zeta) B_k + \sum_{k=1}^{\infty} u_k(\sigma_k\rho, \sigma_k\zeta) C_k + \sum_{k=1}^{\infty} u_k(\omega_k\rho, \omega_k\zeta) D_k$$
$$w = \sum_{k=1}^{2} w_k(\gamma_k\rho, \gamma_k\zeta) B_k + \sum_{k=1}^{\infty} w_k(\sigma_k\rho, \sigma_k\zeta) C_k + \sum_{k=1}^{\infty} w_k(\omega_k\rho, \omega_k\zeta) D_k \qquad (4.3)$$

$$\sigma_z = \sum_{k=1}^{2} \sigma_{zk}(\gamma_k\rho, \gamma_k\zeta) B_k + \sum_{k=1}^{\infty} \sigma_{zk}(\sigma_k\rho, \sigma_k\zeta) C_k + \sum_{k=1}^{\infty} \sigma_{zk}(\omega_k\rho, \omega_k\zeta) D_k$$
$$\sigma_\theta = \sum_{k=1}^{2} \sigma_{\theta k}(\gamma_k\rho, \gamma_k\zeta) B_k + \sum_{k=1}^{\infty} \sigma_{\theta k}(\sigma_k\rho, \sigma_k\zeta) C_k + \sum_{k=1}^{\infty} \sigma_{\theta k}(\omega_k\rho, \omega_k\zeta) D_k$$
$$\sigma_r = \sum_{k=1}^{2} \sigma_{rk}(\gamma_k\rho, \gamma_k\zeta) B_k + \sum_{k=1}^{\infty} \sigma_{rk}(\sigma_k\rho, \sigma_k\zeta) C_k + \sum_{k=1}^{\infty} \sigma_{rk}(\omega_k\rho, \omega_k\zeta) D_k \qquad (4.4)$$
$$\tau_{rz} = \sum_{k=1}^{2} \tau_{rzk}(\gamma_k\rho, \gamma_k\zeta) B_k + \sum_{k=1}^{\infty} \tau_{rzk}(\sigma_k\rho, \sigma_k\zeta) C_k + \sum_{k=1}^{\infty} \tau_{rzk}(\omega_k\rho, \omega_k\zeta) D_k$$

Here B_k, C_k and D_k are constants to be determined; γ_k are roots of the second group; σ_k and ω_k are roots of the third group having expansions whose first terms are given, respectively, as

$$(\sin 2\sigma_{0k} - 2\sigma_{0k}) / \sigma_{0k}^3 = 0, \qquad (\sin 2\omega_{0k} + 2\omega_{0k}) / \omega_{0k} = 0.$$

To determine the coefficients B_k, C_k and D_k, we make use of Lagrange's principle of virtual displacements, using the above terms as generalized displacements.

In the case at hand, the solutions of the homogeneous equations satisfy exactly the equilibrium equations and the boundary conditions on Γ_2, so that the principle of virtual displacements yields:

$$\int_{R_1}^{R_2} (\sigma_z \delta w + \tau_{rz}\delta u)\, r dr = \int_{R_1}^{R_2} (\sigma_{z1}^* \delta w + \tau_{rz1}\delta u)\, r dr \qquad (4.5)$$

The preceding equation yields an infinite system of linear algebraic equations

$$\sum_{p=1}^{2} B_p I(\gamma_k, \gamma_p) + \sum_{p=1}^{\infty} C_p I(\gamma_k, \sigma_p) + \sum_{p=1}^{\infty} D_p I(\gamma_k, \omega_p) = \frac{1}{2G} T(\gamma_k)$$
$$(k = 1, 2)$$
$$\sum_{p=1}^{2} B_p I(\sigma_k, \gamma_p) + \sum_{p=1}^{\infty} C_p I(\sigma_k, \sigma_p) + \sum_{p=1}^{\infty} D_p I(\sigma_k, \omega_p) = \frac{1}{2G} T(\sigma_k) \qquad (4.6)$$
$$(k = 1, 2, \ldots, \infty)$$
$$\sum_{p=1}^{2} B_p I(\omega_k, \gamma_p) + \sum_{p=1}^{\infty} C_p I(\omega_k, \sigma_p) + \sum_{p=1}^{\infty} D_p I(\omega_k, \omega_p) = \frac{1}{2G} T(\omega_k)$$
$$(k = 1, 2, \ldots, \infty)$$

Equations (4.6) contain the following notation:

$$I(\gamma_k, \gamma_s) = Q(\gamma_k, \gamma_s)\,[(\gamma_k^2 - \gamma_s^2)(\nu - 1) M_{k1} M_{s1} + \\ + \gamma_k \gamma_s \rho\,(\gamma_k M_{k1} M_{s0} - \gamma_s M_{s1} M_{k0})]\Big|_{\rho=1}^{\rho=1+\varepsilon} \qquad (\gamma_k \neq \gamma_s)$$

$$I(\gamma_k, \gamma_k) = -{}^1/_2 \gamma_k^{-1} \operatorname{sh} 2\gamma_k l_0 \,\{M_{k0}^2 [(\nu - 1)\rho^2 \gamma_k^2 - {}^4/_3 \rho^4 \gamma_k^4] + \\ + M_{k1}^2 [-3(\nu - 1)\rho^2 \gamma_k^2 - 2(\nu - 1)^2 + {}^2/_3 \rho^2 \gamma_k^2 - {}^4/_3 \rho^4 \gamma_k^4] + \\ + M_{k0} M_{k1} [-4(\nu - 1)\rho\gamma_k - {}^2/_3 \rho^3 \gamma_k^3]\}\Big|_{\rho=1}^{\rho=1+\varepsilon} \qquad (\gamma_k = \gamma_s)$$

$$T(\gamma_k) = \int_{R_1}^{R_2} (\sigma_{z1}^* w_k(\gamma_k \rho, \gamma_k l_0) + \tau_{rz1} u_k(\gamma_k \rho, \gamma_k l_0))\, r\, dr$$

where

$$Q(\gamma_k, \gamma_s) = -\frac{4}{(\gamma_k^2 - \gamma_s^2)^3}\,[(\nu - 1)(\gamma_k^2 - \gamma_s^2)(\gamma_k \sinh \gamma_k l_0 \cosh \gamma_s l_0 - \\ - \gamma_s \cosh \gamma_k l_0 \sinh \gamma_s l_0) + (\gamma_k^2 + \gamma_s^2)(\gamma_k \sinh \gamma_k l_0 \cosh \gamma_s l_0 + \gamma_s \cosh \gamma_k l_0 \sinh \gamma_s l_0)] \qquad (4.7)$$

$$M_{k1} = A_{2k} J_1(\gamma_k \rho) + A_{4k} Y_1(\gamma_k \rho), \quad M_{k0} = A_{2k} Y_0(\gamma_k \rho) + A_{4k} Y_0(\gamma_k \rho), \quad l_0 = l / R_1$$

It may be shown that this system of linear algebraic equations is associated with a positive definite potential energy form ~ , and therefore, for physically meaningful conditions, always has a solution.

We will now investigate the structure of the system under consideration for $\varepsilon \to 0$.

For this purpose, we will expand the coefficients in powers of ε. Thus,

$$I(\gamma_k, \gamma_p) = \varepsilon \sqrt{\varepsilon} I_0(\gamma_k, \gamma_p) + \varepsilon^2 I_1(\gamma_k, \gamma_p) + \varepsilon^2 \sqrt{\varepsilon} I_2(\gamma_k, \gamma_p) + \cdots \qquad (4.8)$$
$$I(\gamma_k, \sigma_p) = \varepsilon^2 I_0(\gamma_k, \sigma_p) + \varepsilon^2 \sqrt{\varepsilon} I_1(\gamma_k, \sigma_p) + \varepsilon^3 I_2(\gamma_k, \sigma_p) + \cdots$$
$$I(\gamma_k, \omega_p) = \varepsilon^2 \sqrt{\varepsilon} I_0(\gamma_k, \omega_p) + \varepsilon^3 I_1(\gamma_k, \omega_p) + \varepsilon^3 \sqrt{\varepsilon} I_2(\gamma_k, \omega_p) + \cdots$$
$$I(\sigma_k, \sigma_p) = \varepsilon^2 I_0(\sigma_k, \sigma_p) + \varepsilon^3 I_1(\sigma_k, \sigma_p) + \varepsilon^4 I_2(\sigma_k, \sigma_p) + \cdots$$
$$I(\sigma_k, \omega_p) = \varepsilon^3 I_0(\sigma_k, \omega_p) + \varepsilon^4 I_1(\sigma_k, \omega_p) + \varepsilon^5 I_2(\sigma_k, \omega_p) + \cdots$$

$$I(\omega_k,\ \omega_p)=\varepsilon^2 I_0(\omega_k,\ \omega_p)+\varepsilon^3 I_1(\omega_k,\ \omega_p)+\varepsilon^4 I_2(\omega_k,\ \omega_p)+\cdots \qquad (4.8)\ \text{cont.}$$

$$T(\gamma_k)=\varepsilon\sqrt{\varepsilon}\,T_0(\gamma_k)+\varepsilon^2 T_1(\gamma_k)+\varepsilon^2\sqrt{\varepsilon}\,T_2(\gamma_k)+\cdots$$

$$T(\sigma_k)=\varepsilon^2 T_0(\sigma_k)+\varepsilon^3 T_1(\sigma_k)+\varepsilon^4 T_2(\sigma_k)+\cdots$$

$$T(\omega_k)=\varepsilon^2 T_0(\omega_k)+\varepsilon^3 T_1(\omega_k)+\varepsilon^4 T_2(\omega_k)+\cdots$$

$$B_k=B_{0k}+\sqrt{\varepsilon}B_{1k}+\varepsilon B_{2k}+\cdots$$

$$C_k=C_{0k}+\sqrt{\varepsilon}C_{1k}+\varepsilon C_{2k}+\cdots$$

$$D_k=D_{0k}+\sqrt{\varepsilon}D_{1k}+\varepsilon D_{2k}+\cdots$$

Here (4.9)

$$\underset{(k\neq p)}{I_0(\gamma_k,\gamma_p)}=-16(\nu-1)(1-\nu^2)\sqrt[4]{3(1-\nu^2)}\left(\sin 2\sqrt[4]{3(1-\nu^2)}\frac{l_0}{\sqrt{\varepsilon}}+\right.$$
$$\left.+\sinh 2\sqrt[4]{3(1-\nu^2)}\frac{l_0}{\sqrt{\varepsilon}}\right)$$

$$\underset{(k=p)}{I_0(\gamma_k,\gamma_k)}=0,\qquad \underset{(k=p)}{I_0(\sigma_k,\sigma_k)}=4\sigma_{0k}^3\left(1-\frac{2}{3}\cos\sigma_{0k}^2\right)\sinh 4\sigma_{0k}\frac{l_0}{\varepsilon}$$

$$I_0(\gamma_k,\sigma_p)=96\nu(1-\nu^2)\frac{1}{\gamma_{0k}\sigma_{0p}}\sin^2\sigma_{0p}\sinh 2\sigma_{0p}\frac{l_0}{\varepsilon}\cosh\gamma_{0k}\frac{l_0}{\sqrt{\varepsilon}}$$

$$\underset{(k\neq p)}{I_0(\sigma_k,\sigma_p)}=\frac{32\sigma_{0k}^2\sigma_{0p}^2}{(\sigma_{0k}^2-\sigma_{0p}^2)^3}\left[(\nu-1)(\sigma_{0k}^2-\sigma_{0p}^2)\left(\sigma_{0k}\sinh 2\sigma_{0k}\frac{l_0}{\varepsilon}\cosh 2\sigma_{0p}\frac{l_0}{\varepsilon}-\right.\right.$$
$$\left.-\sigma_{0p}\,\mathrm{sh}\,2\sigma_{0k}\frac{l_0}{\varepsilon}\cosh 2\sigma_{0k}\frac{l_0}{\varepsilon}\right)+(\sigma_{0k}^2+\sigma_{0p}^2)\times$$
$$\left.\times\left(\sigma_{0k}\sinh 2\sigma_{0k}\frac{l_0}{\varepsilon}\cosh 2\sigma_{0p}\frac{l_0}{\varepsilon}+\sigma_{0p}\sinh 2\sigma_{0p}\frac{l_0}{\varepsilon}\cosh 2\sigma_{0k}\frac{l_0}{\varepsilon}\right)\right](\cos^2\sigma_{0k}-\cos^2\sigma_{0p})$$

$$\underset{(k\neq p)}{I_0(\omega_k,\omega_p)}=\frac{32\omega_{0k}^2\omega_{0p}^2}{(\omega_{0k}^2-\omega_{0p}^2)^3}\left[(\nu-1)(\omega_{0k}^2-\omega_{0p}^2)\left(\omega_{0k}\sinh 2\omega_{0k}\frac{l_0}{\varepsilon}\cosh 2\omega_{0p}\frac{l_0}{\varepsilon}-\right.\right.$$
$$\left.-\omega_{0p}\sinh 2\omega_{0p}\frac{l_0}{\varepsilon}\cosh 2\omega_{0k}\frac{l_0}{\varepsilon}\right)+(\omega_{0k}^2+\omega_{0p}^2)\times$$
$$\left.\times\left(\omega_{0k}\sinh 2\omega_{0k}\frac{l_0}{\varepsilon}\cosh 2\omega_{0p}\frac{l_0}{\varepsilon}+\omega_{0p}\sinh 2\omega_{0p}\frac{l_0}{\varepsilon}\cosh 2\omega_{0k}\frac{l_0}{\varepsilon}\right)\right](\sin^2\omega_{0k}-\sin^2\omega_{0p})$$

$$\underset{(k=p)}{I_0(\omega_k,\omega_k)}=4\omega_{0k}^3\left(1-\frac{2}{3}\sin^2\omega_{0k}\right)\sinh 4\omega_{0k}\frac{l_0}{\varepsilon}$$

$$T_0(\gamma_k)=\int_{-1}^{1}\left\{\sigma_{z1}^*\left[4(\nu-1)\nu+2\gamma_{0k}^2(\nu-1)r_0\right]\sinh\gamma_{0k}\frac{l_0}{\sqrt{\varepsilon}}-\right.$$
$$\left.-4(\nu-1)\tau_{rz1}\gamma_{0k}\cosh\gamma_{0k}\frac{l_0}{\sqrt{\varepsilon}}\right\}dr_0$$

$$T_0(\sigma_k)=\int_{-1}^{1}\left[\sigma_{z1}^* w_{0k}(\sigma_{0k})\sinh 2\sigma_{0k}\frac{l_0}{\varepsilon}+\tau_{rz1}u_{0k}(\sigma_{0k})\,2\sigma_{0k}\cosh 2\sigma_{0k}\frac{l_0}{\varepsilon}\right]dr_0 \qquad (4.10)$$

$$T_0(\omega_k)=\int_{-1}^{1}\left[\sigma_{z1}^* w_{0k}(\omega_{0k})\sinh 2\omega_{0k}\frac{l_0}{\varepsilon}+\tau_{rz1}u_{0k}(\omega_{0k})\,2\omega_{0k}\cosh 2\omega_{0k}\frac{l_0}{\varepsilon}\right]dr_0$$

An analysis of the structure of the system for $\varepsilon \to 0$ leads to the conclusion that the first approximations of each group of coefficients B_{0k}, C_{0k} and D_{0k} may be determined independently od each other, i.e. B_{0k} may be found from two equations; C_{0k} may be obtained from a countably infinite system of algebraic equations, and D_{0k} may be obtained separately from another countably infinite system

$$\sum_{p=1}^{2}\left(B_{0p}I_0(\gamma_k,\gamma_p)-\frac{1}{2G}T_0(\gamma_k)\right)=0 \qquad (k=1,2) \tag{4.11}$$

$$\sum_{p=1}^{\infty}\left(C_{0p}I_0(\sigma_k,\sigma_p)-\frac{1}{2G}T_0(\sigma_k)\right)=0 \qquad (k=1,2,\ldots,\infty) \tag{4.12}$$

$$\sum_{p=1}^{\infty}\left(D_{0p}I_0(\omega_k,\omega_p)-\frac{1}{2G}T_0(\omega_k)\right)=0 \qquad (k=1,2,\ldots,\infty) \tag{4.13}$$

It should be emphasized that the determinations of B_{1k}, C_{1k} and D_{1k} invariably lead to inversion of matrices associated with Equations (4.11) to (4.13). The elements of these matrices are independent of the type of loading applied on the end faces Γ_1, so that the inversion need only be carried out once. For a semi-infinite cylinder, $m_k = \exp(-\gamma_k\zeta)$, the system of equations is similar to Equations (4.11) to (4.13), but the expressions for I_0 and T_0 are different

$$I_0\underset{(k\neq p)}{(\gamma_k,\gamma_p)} = 32(1-\nu^2)(\nu-1)\sqrt[4]{3(1-\nu^2)}\exp\left(-2\sqrt[4]{3(1-\nu^2)}\frac{l_0}{\sqrt{\varepsilon}}\right)$$

$$I_0\underset{(k=p)}{(\gamma_k,\gamma_k)}=0$$

$$I_0\underset{(k=p)}{(\sigma_k,\sigma_k)} = -4\sigma_{0k}^3\left(1-\frac{2}{3}\cos^2\sigma_{0k}\right)\exp\left(-4\sigma_{0k}\frac{l_0}{\varepsilon}\right)$$

$$I_0\underset{(k\neq p)}{(\sigma_k,\sigma_p)} = -\frac{32\sigma_{0k}^2\sigma_{0p}^2}{(\sigma_{0k}^2-\sigma_{0p}^2)^2(\sigma_{0k}-\sigma_{0p})}[\nu(\sigma_{0k}-\sigma_{0p})^2+2\sigma_{0k}\sigma_{0p}]\times$$
$$\times(\cos^2\sigma_{0k}-\cos^2\sigma_{0p})\exp\left[-(\sigma_{0k}+\sigma_{0p})\frac{l_0}{\varepsilon}\right]$$

$$I_0\underset{(k\neq p)}{(\omega_k,\omega_p)} = -\frac{32\omega_{0k}^2\omega_{0p}^2}{(\omega_{0k}^2-\omega_{0p}^2)^2(\omega_{0k}-\omega_{0p})}[\nu(\omega_{0k}-\omega_{0p})^2+2\omega_{0k}\omega_{0p}]\times$$
$$\times(\sin^2\omega_{0k}-\sin^2\omega_{0p})\exp\left\{-(\omega_{0k}+\omega_{0p})\frac{l_0}{\varepsilon}\right\}$$

$$I_0\underset{(k=p)}{(\omega_k,\omega_k)} = -4\omega_{0k}^3\left(1-\frac{2}{3}\sin^2\omega_{0k}\right)\exp\left(-4\omega_{0k}\frac{l_0}{\varepsilon}\right) \tag{4.14}$$

$$T_0(\gamma_k) =$$
$$=\int_{-1}^{1}\{\sigma_{z1}^*[4(\nu-1)\nu+2\gamma_{0k}^2(\nu-1)r_0]+4(\nu-1)\tau_{rz1}\gamma_{0k}\}\exp\left(-\gamma_{0k}\frac{l_0}{\sqrt{\varepsilon}}\right)dr_0$$

$$T_0(\sigma_k) = \int_{-1}^{1} [\sigma_{z1}{}^* w_{0k}(\sigma_{0k}) - \tau_{rz1} u_{0k}(\sigma_{0k})\, 2\sigma_{0k}] \exp\left(-2\sigma_{0k}\frac{l_0}{\varepsilon}\right) dr_0 \qquad (4.15)$$

$$T_0(\omega_k) = \int_{-1}^{1} [\sigma_{z1}{}^* w_{0k}(\omega_{0k}) - \tau_{rz1} u_{0k}(\omega_{0k})\, 2\omega_{0k}] \exp\left(-2\omega_{0k}\frac{l_0}{\varepsilon}\right) dr_0$$

It should be noted that Equations (4.12) and (4.13), using the definitions given in (4.14) and (4.15) arise in plate theory; they may be solved by truncation. Equations (4.12) are associated with bending, whereas Equations (4.13) are associated with the extension of a plate.

5. Balancing of stresses on the cylindrical boundary surfaces. Construction of improved theories. So far we have investigated the homogeneous equations. However, the approach used in Section 1 permits the construction of a solution if the cylindrical surface is loaded as well, provided that the load is expandable in a Fourier series. To illustrate this approach, we will find the solution corresponding to the kth harmonic of the external load and satisfying the following boundary conditions:

$$\sigma_r(R_1, z) = 2GA\cos k\zeta, \qquad \tau_{rz}(R_1, z) = 2GB\sin k\zeta$$
$$\sigma_r(R_2, z) = 2GC\cos k\zeta, \qquad \tau_{rz}(R_2, z) = 2GD\sin k\zeta \qquad (5.1)$$

This problem is of an auxilliary character; its solution may be written in the form

$$(R_1/R_2)\,\gamma^2\Theta(\gamma,\varepsilon)\,u = (AP_1(\rho,\gamma,\varepsilon) + BP_2(\rho,\gamma,\varepsilon) + CP_3(\rho,\gamma,\varepsilon) + DP_4(\rho,\gamma,\varepsilon))\cos k\zeta \qquad (5.2)$$

$$(R_1/R_2)\,\gamma^2\Theta(\gamma,\varepsilon)\,w = (AQ_1(\rho,\gamma,\varepsilon) + BQ_2(\rho,\gamma,\varepsilon) + CQ_3(\rho,\gamma,\varepsilon) + DQ_4(\rho,\gamma,\varepsilon))\sin k\zeta \qquad (5.3)$$

where $\gamma = tk$, Θ is defined in (2.1) and

$$P_i = A_{1i}J_1(\gamma\rho) - A_{2i}\gamma\rho J_0(\gamma\rho) + A_{3i}Y_1(\gamma\rho) - A_{4i}\gamma\rho Y_0(\gamma\rho) \qquad (5.4)$$

$$Q_i = -A_{1i}J_0(\gamma\rho) + A_{2i}[4(1-\nu)J_0(\gamma\rho) - \gamma\rho J_1(\gamma\rho)] - A_{3i}Y_0(\gamma\rho) + A_{4i}[4(1-\nu)Y_0(\gamma\rho) - \gamma\rho Y_1(\gamma\rho)] \qquad (5.5)$$

The quantities A_{ki} are obtained from Equations (5.4) to (5.7). Setting $\xi_1 = \eta$, $\xi_2 = \eta(1+\varepsilon)$, we obtain

$$A_{11} = \gamma^3\{[\xi_2 + 2(\nu-1)\xi_2^{-1}]Y_1(\xi_2)[\xi_1 L_{01} - 2(\nu-1)L_{11}] + \xi_2 Y_0(\xi_2)[\xi_1 L_{00} - 2(\nu-1)L_{10}] + 2(\nu-1)\xi_1\xi_2^{-1}Y_0(\xi_1) + [\xi_2 + 2(\nu-1)\xi_2^{-1}]Y_1(\xi_1)\} \qquad (5.6)$$

$$A_{21} = \gamma^3\{[\xi_2 + 2(\nu-1)\xi_2^{-1}]Y_1(\xi_2)L_{11} + \xi_2 Y_0(\xi_2)L_{10} - \xi_1\xi_2^{-1}Y_0(\xi_1)\}$$

$$A_{31} = -\gamma^3\{[\xi_2 + 2(\nu-1)\xi_2^{-1}]J_1(\xi_2)[\xi_1 L_{01} - 2(\nu-1)L_{11}] + \xi_2 J_0(\xi_2)[\xi_1 L_{00} - 2(\nu-1)L_{10}] + 2(\nu-1)\xi_1\xi_2^{-1}J_0(\xi_1) + [\xi_2 + 2(\nu-1)\xi_2^{-1}]J_1(\xi_1)\}$$

$$A_{41} = -\gamma^3\{[\xi_2 + 2(\nu-1)\xi_2^{-1}]J_1(\xi_2)L_{11} + \xi_2 J_0(\xi_2)L_{10} - \xi_1\xi_2^{-1}J_0(\xi_1)\}$$

$$A_{12} = \gamma^3\{Y_1(\xi_2)[\xi_2 + 2(\nu-1)\xi_2^{-1}][(2\nu-1)L_{01} + \xi_1 L_{11}] + \xi_2 Y_0(\xi_2)[(2\nu-1)L_{00} + \xi_1 L_{10}] + 2(\nu-1)\frac{1}{\xi_2}Y_1(\xi_1)[\xi_1 + (2\nu-1)\xi_1^{-1}] - \xi_2 Y_0(\xi_1) + \xi_2\xi_1^{-1}Y_1(\xi_1)\}$$

$$A_{22} = \gamma^3\{Y_1(\xi_2)[\xi_2 + 2(\nu-1)\xi_2^{-1}](\xi_1^{-1}L_{11} - L_{01}) + \xi_2 Y_0(\xi_2)(\xi_1^{-1}L_{10} - L_{00}) - \xi_2^{-1}Y_1(\xi_1)[\xi_1 + 2(\nu-1)\xi_1^{-1}] - \xi_2^{-1}Y_0(\xi_1)\}$$

$$A_{32} = -\gamma^3 \{J_1(\xi_2)[\xi_2 + 2(\nu-1)\xi_2^{-1}][(2\nu-1)L_{01} + \xi_1 L_{11}] + \xi_2 J_0(\xi_2)[\xi_1 L_{10} + (2\nu-1)L_{00}] + 2(\nu-1)\xi_2^{-1}J_1(\xi_1)[\xi_1 + (2\nu-1)\xi_1^{-1}] - \xi_2 J_0(\xi_1) + \xi_2\xi_1^{-1}J_1(\xi_1)\}$$

$$A_{42} = -\gamma^3 \{J_1(\xi_2)[\xi_2 + 2(\nu-1)\xi_2^{-1}](\xi_1^{-1}L_{11} - L_{01}) + \xi_2 J_0(\xi_2)(\xi_1^{-1}L_{10} - L_{00}) - \xi_2^{-1}J_1(\xi_1)[\xi_1 + 2(\nu-1)\xi_1^{-1}] - \xi_2^{-1}J_0(\xi_1)\}$$

$$A_{13} = \gamma^3 \{[\xi_1 + 2(\nu-1)\xi_1^{-1}]Y_1(\xi_1)[2(\nu-1)L_{11} - \xi_2 L_{10}] + \xi_1 Y_0(\xi_1) \times [2(\nu-1)L_{01} - \xi_2 L_{00}] + 2(\nu-1)\xi_2\xi_1^{-1}Y_0(\xi_2) + [\xi_1 + 2(\nu-1)\xi_1^{-1}]Y_1(\xi_2)\}$$

$$A_{23} = -\gamma^3 \{[\xi_1 + 2(\nu-1)\xi_1^{-1}]Y_1(\xi_1)L_{11} + \xi_1 Y_0(\xi_1)L_{01} + \xi_2\xi_1^{-1}Y_0(\xi_2)\}$$

$$A_{33} = -\gamma^3 \{[\xi_1 + 2(\nu-1)\xi_1^{-1}]J_1(\xi_1)[2(\nu-1)L_{11} - \xi_2 L_{10}] + \xi_1 J_0(\xi_1) \times [2(\nu-1)L_{01} - \xi_2 L_{00}] + 2(\nu-1)\xi_1^{-1}\xi_2 J_0(\xi_2) + [\xi_1 + 2(\nu-1)\xi_1^{-1}]J_1(\xi_2)\}$$

$$A_{43} = \gamma^3 \{[\xi_1 + 2(\nu-1)\xi_1^{-1}]J_1(\xi_1)L_{11} + \xi_1 J_0(\xi_1)L_{01} + \xi_2\xi_1^{-1}J_0(\xi_2)\}$$

$$A_{14} = -\gamma^3 \{Y_1(\xi_1)[\xi_1 + 2(\nu-1)\xi_1^{-1}][(2\nu-1)L_{10} + \xi_2 L_{11}] + \xi_1 Y_0(\xi_1) \times [(2\nu-1)L_{00} + \xi_2 L_{01}] - 2(\nu-1)\xi_1^{-1}Y_1(\xi_2)[\xi_2 + (2\nu-1)\xi_2^{-1}] + \xi_1 Y_0(\xi_2) - \xi_1\xi_2^{-1}Y_1(\xi_2)\}$$

$$A_{24} = \gamma^3 \{Y_1(\xi_1)[\xi_1 + 2(\nu-1)\xi_1^{-1}](L_{10} - \xi_2^{-1}L_{11}) + \xi_1 Y_0(\xi_1)(L_{00} - \xi_2^{-1}L_{01}) - \xi_1^{-1}Y_1(\xi_2)[\xi_2 + 2(\nu-1)\xi_2^{-1}] - \xi_1^{-1}Y_0(\xi_2)\}$$

$$A_{34} = \gamma^3 \{J_1(\xi_1)[\xi_1 + 2(\nu-1)\xi_1^{-1}][(2\nu-1)L_{10} + \xi_2 L_{11}] + \xi_1 J_0(\xi_1)[\xi_2 L_{01} + (2\nu-1)L_{00}] - 2(\nu-1)\xi_1^{-1}J_1(\xi_2)[\xi_2 + (2\nu-1)\xi_2^{-1}] + \xi_1 J_0(\xi_2) - \xi_1\xi_2^{-1}J_1(\xi_2)\}$$

$$A_{44} = -\gamma^3 \{J_1(\xi_1)[\xi_1 + 2(\nu-1)\xi_1^{-1}](L_{10} - \xi_2^{-1}L_{11}) + \xi_1 J_0(\xi_1)(L_{00} - \xi_2^{-1}L_{01}) - \xi_1^{-1}J_1(\xi_2)[\xi_2 + 2(\nu-1)\xi_2^{-1}] - \xi_1^{-1}J_0(\xi_2)\} \qquad (5.7)$$

The exact solution thus obtained will be used to evaluate the accuracy of applied theories. Suppose that we are interested in some characteristic of the preceding solution. For example, suppose that we are interested in the behavior of u and w on the middle surface $r = 0.5(R_1 + R_2)$ when the relative thickness of the shell $\epsilon \to 0$. To determine this behavior, we expand θ, P_i and Q_i in power series of ϵ. Retaining terms up to a given power of ϵ in both the left and right sides of Equations (5.2) and (5.3), we can, for the given loading on the cylindrical boundary surface, obtain relations of the form

$$u \sum_{p=1}^{N} \Delta_p^*(ik)\varepsilon^p = \sum_{p=1}^{N} (AP_{1p}(ik) + BP_{2p}(ik) + CP_{3p}(ik) + DP_{4p}(ik))\varepsilon^p \cos k\zeta$$
$$w \sum_{p=1}^{N} \Delta_p^*(ik)\varepsilon^p = \sum_{p=1}^{N} (AQ_{1p}(ik) + BQ_{2p}(ik) + CQ_{3p}(ik) + DQ_{4p}(ik))\varepsilon^p \sin k\zeta \qquad (5.8)$$

where $\Delta_p^*(ik)$, $P_{ip}(ik)$ and $Q_{ip}(ik)$ are polynomials in ik.

The smaller ϵk becomes and the larger we make N, the more accurate Equations (5.8) will be. It is readily seen that Equations (5.8) may be obtained if we assume that u and w are found by means of some shell theory which is given by Equations

$$\sum_{p=1}^{N} \Delta_p^*\left(\frac{d}{d\zeta}\right) u\varepsilon^p = \sum_{p=1}^{N} \left(P_{1p}\left(\frac{d}{d\zeta}\right)\sigma_r^*(R_1, z) + P_{2p}\left(\frac{d}{d\zeta}\right)\tau_{rz}^*(R_1, z) + P_{3p}\left(\frac{d}{d\zeta}\right)\sigma_r^*(R_2, z) + P_{4p}\left(\frac{d}{d\zeta}\right)\tau_{rz}^*(R_2, z)\right)\varepsilon^p \qquad (5.9)$$

$$\sum_{p=1}^{N} \Delta_p^*\left(\frac{d}{d\zeta}\right) w\varepsilon^p = \sum_{p=1}^{N}\Big(Q_{1p}\left(\frac{d}{d\zeta}\right)\sigma_r^*(R_1, z) + Q_{2p}\left(\frac{d}{d\zeta}\right)\tau_{rz}^*(R_1, z) + Q_{3p}\left(\frac{d}{d\zeta}\right)\sigma_r^*(R_2, z) + Q_{4p}\left(\frac{d}{d\zeta}\right)\tau_{rz}^*(R_2, z)\Big)\varepsilon^p \tag{5.9 cont.}$$

Clearly, the smaller εk becomes the more accurate Equations (5.8) and (5.9) will become, where k represents the end of the essential part of the spectrum of external loading. Thus, we have developed a practical approach to the construction of applied theories for cylindrical shells. Moreover, if more terms are retained in Equations (5.9) we will obtain a more accurate theory. Note that the preceding applied theories are intended only for the balancing of stresses on the cylindrical boundary surfaces.

The balancing of stresses on the end faces is accomplished by the method previously discussed in connection with the homogeneous equations. Nevertheless, the problem arises concerning the relationship between the edge effects of the applied theories, Equations (5.9), and the exact edge effects obtained from the characteristic equation (2.1). Thus, if we seek a complimentary solution of Equations (5.9) in the form $u, w \sim e^{\gamma\zeta}$, we obtain the following equation in γ :

$$P_N(\gamma) = \Delta_1^*(\gamma)\varepsilon + \Delta_2^*(\gamma)\varepsilon^2 + \ldots + \Delta_N^*(\gamma)\varepsilon^N = 0 \tag{5.10}$$

From Equation (5.10) it is not difficult to find the first $[\frac{1}{2}N]$ terms in the series expansion of the roots of the second group. The roots of the third group which are associated with the St.Venant edge effects can not be determined from Equation (5.10).

As a specific example of an applied theory based on Equations (5.9), we develop the theory for $N = 4$. Thus

$$\Big\{\varepsilon^2\Big[4(\nu^2-1)\frac{d^2}{d\zeta^2}\Big] + \varepsilon^3\Big[-4(\nu^2-1)\frac{d^2}{d\zeta^2}\Big] + \varepsilon^4\Big[-\frac{1}{3}\frac{d^6}{d\zeta^6} - \frac{4}{3}(\nu^2-1)\frac{d^4}{d\zeta^4} + 5(\nu^2-1)\frac{d^2}{d\zeta^2}\Big]\Big\} u = \frac{R_1}{2G}\Big\{\Big\langle\varepsilon\Big[-4(\nu-1)\frac{d^2}{d\zeta^2}\Big] + \varepsilon^2\Big[-2(\nu-1)^2\frac{d^2}{d\zeta^2}\Big] +$$

$$+ \varepsilon^3\Big[-\frac{1}{2}\frac{d^4}{d\zeta^4} + \frac{5}{6}(\nu-1)\frac{d^4}{d\zeta^4} + \frac{5}{2}(\nu-1)^2\frac{d^2}{d\zeta^2}\Big] + \varepsilon^4\Big[-\frac{1}{4}(\nu-1)\frac{d^4}{d\zeta^4} + \frac{5}{12}(\nu-1)^2 \times \frac{d^4}{d\zeta^4} - 3(\nu-1)^2\frac{d^2}{d\zeta^2}\Big]\Big\rangle\sigma_r^*(R_1, z) + \Big\langle\varepsilon\Big[-4\nu(\nu-1)\frac{d}{d\zeta}\Big] +$$

$$+ \varepsilon^2\Big[4\nu(\nu-1)\frac{d}{d\zeta} + 2(\nu-1)\frac{d^3}{d\zeta^3}\Big] + \varepsilon^3\Big[-\frac{1}{2}\frac{d^3}{d\zeta^3} - 5\nu(\nu-1)\frac{d}{d\zeta} + \frac{7}{6}(\nu-1)^2\frac{d^3}{d\zeta^3}\Big] +$$

$$+ \varepsilon^4\Big[\frac{1}{12}\frac{d^5}{d\zeta^5} + 6\nu(\nu-1)\frac{d}{d\zeta} - \frac{7}{12}(\nu-1)\frac{d^3}{d\zeta^3} - \frac{1}{4}(\nu-1)\frac{d^5}{d\zeta^5} - \frac{19}{12}(\nu-1)^2\frac{d^3}{d\zeta^3}\Big]\Big\rangle \times \tau_{rz}^*(R_1, z) + \Big\langle\varepsilon\Big[4(\nu-1)\frac{d^2}{d\zeta^2}\Big] + \varepsilon^2\Big[-2(\nu-1)^2\frac{d^2}{d\zeta^2}\Big] +$$

$$+ \varepsilon^3\Big[\frac{1}{2}\frac{d^4}{d\zeta^4} - \frac{5}{6}(\nu-1)\frac{d^4}{d\zeta^4} - \frac{1}{2}(\nu-1)^2\frac{d^2}{d\zeta^2}\Big] +$$

$$+ \varepsilon^4\Big[-\frac{1}{4}(\nu-1)\frac{d^4}{d\zeta^4} + \frac{5}{12}(\nu-1)^2\frac{d^4}{d\zeta^4}\Big]\Big\rangle\sigma_r^*(R_2, z) + \Big\langle\varepsilon\Big[4\nu(\nu-1)\frac{d}{d\zeta}\Big] +$$

$$+ \varepsilon^2\Big[2(\nu-1)\frac{d^3}{d\zeta^3}\Big] + \varepsilon^3\Big[\frac{1}{2}\frac{d^3}{d\zeta^3} + \nu(\nu-1)\frac{d}{d\zeta} - \frac{7}{6}(\nu-1)^2\frac{d^3}{d\zeta^3}\Big] +$$

$$+ \varepsilon^4\Big[-\frac{1}{2}\frac{d^3}{d\zeta^3} - \nu(\nu-1)\frac{d}{d\zeta} - \frac{7}{12}(\nu-1)\frac{d^3}{d\zeta^3} - \frac{5}{12}(\nu-1)^2\frac{d^3}{d\zeta^3} + \frac{1}{12}\frac{d^5}{d\zeta^5} - \frac{1}{4}(\nu-1)\frac{d^5}{d\zeta^5}\Big]\Big\rangle\tau_{rz}^*(R_2, z)\Big\}$$

$$\Big\{\varepsilon^2\Big[4(\nu^2-1)\frac{d^2}{d\zeta^2}\Big]+\varepsilon^3\Big[-4(\nu^2-1)\frac{d^2}{d\zeta^2}\Big]+\varepsilon^4\Big[-\frac{1}{3}\frac{d^6}{d\zeta^6}-\frac{4}{3}(\nu^2-1)\frac{d^4}{d\zeta^4}+ +5(\nu^2-1)\frac{d^2}{d\zeta^2}\Big]\Big\}w=\frac{R_1}{2G}\Big\{\Big\langle\varepsilon\Big[4\nu(\nu-1)\frac{d}{d\zeta}\Big]+\varepsilon^2\Big[-6\nu(\nu-1)\frac{d}{d\zeta}\Big]+ +\varepsilon^3\Big[8\nu(\nu-1)\frac{d}{d\zeta}-\frac{3}{2}(\nu-1)\frac{d^3}{d\zeta^3}-\frac{7}{6}(\nu-1)^2\frac{d^3}{d\zeta^3}\Big]+ +\varepsilon^4\Big[-10\nu(\nu-1)\frac{d}{d\zeta}+\frac{1}{6}\nu\frac{d^5}{d\zeta^5}+\frac{25}{12}(\nu-1)\frac{d^3}{d\zeta^3}+\frac{5}{3}(\nu-1)^2\frac{d^3}{d\zeta^3}\Big]\Big\rangle\sigma_r^*(R_1,z)+ +\Big\langle\varepsilon[4(\nu-1)]+\varepsilon^2\Big[-6(\nu-1)-2\nu(\nu-1)\frac{d^2}{d\zeta^2}\Big]+\varepsilon^3\Big[8(\nu-1)+\frac{5}{6}(\nu-1)\frac{d^2}{d\zeta^2}+ +\frac{1}{3}(\nu-1)\frac{d^4}{d\zeta^4}+\frac{11}{6}(\nu-1)^2\frac{d^2}{d\zeta^2}\Big]+\varepsilon^4\Big[-\frac{5}{12}\frac{d^4}{d\zeta^4}-10(\nu-1)-\frac{9}{4}(\nu-1)\frac{d^2}{d\zeta^2}+ +\frac{7}{12}(\nu-1)\frac{d^4}{d\zeta^4}-\frac{29}{12}(\nu-1)^2\frac{d^2}{d\zeta^2}+\frac{5}{12}(\nu-1)^2\frac{d^4}{d\zeta^4}\Big]\Big\rangle\tau_{rz}^*(R_1,z)+$$

$$+\Big\langle\varepsilon\Big[-4\nu(\nu-1)\frac{d}{d\zeta}\Big]+\varepsilon^2\Big[-2\nu(\nu-1)\frac{d}{d\zeta}\Big]+\varepsilon^3\Big[\frac{3}{2}(\nu-1)\frac{d^3}{d\zeta^3}+\frac{7}{6}(\nu-1)^2\frac{d^3}{d\zeta^3}\Big]+ +\varepsilon^4\Big[\frac{\nu}{6}\frac{d^5}{d\zeta^5}+\frac{7}{12}(\nu-1)\frac{d^3}{d\zeta^3}+\frac{1}{2}(\nu-1)^2\frac{d^3}{d\zeta^3}\Big]\Big\rangle\sigma_r^*(R_2,z)+\Big\langle\varepsilon[-4(\nu-1)]+ +\varepsilon^2\Big[2(\nu-1)-2\nu(\nu-1)\frac{d^2}{d\zeta^2}\Big]+\varepsilon^3\Big[-2(\nu-1)+\frac{7}{6}(\nu-1)\frac{d^2}{d\zeta^2}-\frac{1}{3}(\nu-1)\frac{d^4}{d\zeta^4}+ +\frac{1}{6}(\nu-1)^2\frac{d^2}{d\zeta^2}\Big]+\varepsilon^4\Big[\frac{1}{6}\frac{d^4}{d\zeta^4}+2(\nu-1)-\frac{17}{12}(\nu-1)\frac{d^2}{d\zeta^2}+ +\frac{7}{12}(\nu-1)\frac{d^4}{d\zeta^4}-\frac{3}{4}(\nu-1)^2\frac{d^2}{d\zeta^2}+\frac{5}{12}(\nu-1)^2\frac{d^4}{d\zeta^4}\Big]\Big\rangle\tau_{rz}^*(R_2,z) \qquad (5.11)$$

For comparison, (5.12) gives a comparable form of Vlasov's theory, while Novozhilov's theory is given in (5.13).

$$\Big\{\varepsilon^2\Big[4(\nu^2-1)\frac{d^2}{d\zeta^2}\Big]+\varepsilon^3\Big[-4(\nu^2-1)\frac{d^2}{d\zeta^2}\Big]+\varepsilon^4\Big[-\frac{2}{3}\nu\frac{d^4}{d\zeta^4}-\frac{1}{3}\frac{d^6}{d\zeta^6}-\frac{1}{3}\frac{d^2}{d\zeta^2}+ +3(\nu^2-1)\frac{d^2}{d\zeta^2}\Big]\Big\}u=\frac{R_1}{2G}\Big\{\Big\langle\varepsilon\Big[-4(\nu-1)\frac{d^2}{d\zeta^2}\Big]\Big\rangle\sigma_r^*(R_1,z)+\Big\langle\varepsilon\Big[-4\nu(\nu-1)\frac{d}{d\zeta}\Big]+ +\varepsilon^3\Big[\frac{1}{3}(\nu-1)\frac{d^3}{d\zeta^3}\Big]+\varepsilon^4\Big[-\frac{1}{3}(\nu-1)\frac{d^3}{d\zeta^3}\Big]\Big\rangle\tau_{rz}^*(R_1,z)\Big\} \qquad (5.12)$$

$$\Big\{\varepsilon^2\Big[4(\nu^2-1)\frac{d^2}{d\zeta^2}\Big]+\varepsilon^3\Big[-4(\nu^2-1)\frac{d^2}{d\zeta^2}\Big]+\varepsilon^4\Big[-\frac{2}{3}\nu\frac{d^4}{d\zeta^4}-\frac{1}{3}\frac{d^6}{d\zeta^6}-\frac{1}{3}\frac{d^2}{d\zeta^2}+ +3(\nu^2-1)\frac{d^2}{d\zeta^2}\Big]\Big\}w=\frac{R_1}{2G}\Big\{\Big\langle\varepsilon\Big[4\nu(\nu-1)\frac{d}{d\zeta}\Big]+\varepsilon^3\Big[-\frac{1}{3}(\nu-1)\frac{d^3}{d\zeta^3}\Big]+ +\varepsilon^4\Big[\frac{1}{3}(\nu-1)\frac{d^3}{d\zeta^3}\Big]\Big\rangle\sigma_r^*(R_1,z)+\Big\langle\varepsilon[4(\nu-1)]+\varepsilon^3\Big[\frac{1}{3}(\nu-1)\frac{d^4}{d\zeta^4}+ +\frac{1}{3}(\nu-1)\Big]+\varepsilon^4\Big[-\frac{1}{3}(\nu-1)\frac{d^4}{d\zeta^4}-\frac{1}{3}(\nu-1)\Big]\Big\rangle\tau_{rz}^*(R_1,z)\Big\}$$

$$\Big\{\varepsilon^2\Big[4(\nu^2-1)\frac{d^2}{d\zeta^2}\Big]+\varepsilon^3\Big[-4(\nu^2-1)\frac{d^2}{d\zeta^2}\Big]+\varepsilon^4\Big[-\frac{1}{3}\frac{d^6}{d\zeta^6}+3(\nu^2-1)\frac{d^2}{d\zeta^2}\Big]\Big\}u= =\frac{R_1}{2G}\Big\{\Big\langle\varepsilon\Big[4(\nu-1)\frac{d^2}{d\zeta^2}\Big]\Big\rangle\sigma_r^*(R_2,z)+\Big\langle\varepsilon\Big[4\nu(\nu-1)\frac{d}{d\zeta}\Big]\Big\rangle\tau_{rz}^*(R_2,z)\Big\}$$

$$\left\{\varepsilon^2\left[4(\nu^2-1)\frac{d^3}{d\zeta^3}\right]+\varepsilon^3\left[-4(\nu^2-1)\frac{d^3}{d\zeta^3}\right]+\varepsilon^4\left[-\frac{1}{3}\frac{d^7}{d\zeta^7}+3(\nu^2-1)\frac{d^3}{d\zeta^3}\right]\right\}w=$$
$$=\frac{R_1}{2G}\left\{\left\langle\varepsilon\left[-4\nu(\nu-1)\frac{d^2}{d\zeta^2}\right]\right\rangle\sigma_r^*(R_2,z)+\left\langle\varepsilon\left[-4(\nu-1)\frac{d}{d\zeta}\right]+\right.\right.$$
$$\left.\left.+\varepsilon^3\left[-\frac{1}{3}(\nu-1)\frac{d^5}{d\zeta^5}\right]+\varepsilon^4\left[\frac{1}{3}(\nu-1)\frac{d^5}{d\zeta^5}\right]\right\rangle\tau_{rz}^*(R_2,z)\right\} \qquad (5.13)$$

It can be seen that (5.11) coincide with (5.12) and (5.13) only in first order terms.

The same conclusion follows from [7], where approximate differential equations for a cylindrical shell have been obtained in a different form.

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